NAG Toolbox for MATLAB

s21cb

1 Purpose

s21cb evaluates the Jacobian elliptic functions $\operatorname{sn} z$, $\operatorname{cn} z$ and $\operatorname{dn} z$ for a complex argument z.

2 Syntax

```
[sn, cn, dn, ifail] = s21cb(z, ak2)
```

3 Description

s21cb evaluates the Jacobian elliptic functions $\operatorname{sn}(z \mid k)$, $\operatorname{cn}(z \mid k)$ and $\operatorname{dn}(z \mid k)$ given by

$$\begin{aligned} \operatorname{sn}(z \mid k) &= \sin\phi \\ \operatorname{cn}(z \mid k) &= \cos\phi \\ \operatorname{dn}(z \mid k) &= \sqrt{1 - k^2 \sin^2\phi}, \end{aligned}$$

where z is a complex argument, k is a real parameter (the *modulus*) with $k^2 \leq 1$ and ϕ (the *amplitude* of z) is defined by the integral

$$z = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}.$$

The above definitions can be extended for values of $k^2 > 1$ (see Salzer 1962) by means of the formulae

$$\begin{aligned} \operatorname{sn}(z \mid k) &= k_1 \operatorname{sn}(kz \mid k_1) \\ \operatorname{cn}(z \mid k) &= \operatorname{dn}(kz \mid k_1) \\ \operatorname{dn}(z \mid k) &= \operatorname{cn}(kz \mid k_1), \end{aligned}$$

where $k_1 = 1/k$.

Special values include

$$\begin{aligned} \operatorname{sn}(z \mid 0) &= \sin z \\ \operatorname{cn}(z \mid 0) &= \cos z \\ \operatorname{dn}(z \mid 0) &= 1 \\ \operatorname{sn}(z \mid 1) &= \tanh z \\ \operatorname{cn}(z \mid 1) &= \operatorname{sech} z \\ \operatorname{dn}(z \mid 1) &= \operatorname{sech} z. \end{aligned}$$

These functions are often simply written as $\operatorname{sn} z$, $\operatorname{cn} z$ and $\operatorname{dn} z$, thereby avoiding explicit reference to the parameter k. They can also be expressed in terms of Jacobian theta functions (see s21cc).

Another nine elliptic functions may be computed via the formulae

$$\begin{aligned} \operatorname{cd} z &= \operatorname{cn} z / \operatorname{dn} z \\ \operatorname{sd} z &= \operatorname{sn} z / \operatorname{dn} z \\ \operatorname{nd} z &= 1 / \operatorname{dn} z \\ \operatorname{dc} z &= \operatorname{dn} z / \operatorname{cn} z \\ \operatorname{nc} z &= 1 / \operatorname{cn} z \\ \operatorname{sc} z &= \operatorname{sn} z / \operatorname{cn} z \\ \operatorname{ns} z &= 1 / \operatorname{sn} z \\ \operatorname{ds} z &= \operatorname{dn} z / \operatorname{sn} z \\ \operatorname{cs} z &= \operatorname{cn} z / \operatorname{sn} z \end{aligned}$$

(see Abramowitz and Stegun 1972).

The values of $\operatorname{sn} z$, $\operatorname{cn} z$ and $\operatorname{dn} z$ are obtained by calls to s21ca. Further details can be found in Section 8.

4 References

Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (3rd Edition) Dover Publications

Salzer H E 1962 Quick calculation of Jacobian elliptic functions *Comm. ACM* **5** 399

5 Parameters

5.1 Compulsory Input Parameters

1: **z – complex scalar**

The argument z of the functions.

Constraints:

$\mathrm{abs}(\mathrm{Re}(\mathbf{z})) \leq= \sqrt{\lambda}$;
$\mathrm{abs}(\mathrm{Im}(\mathbf{z})) \leq \sqrt{\lambda}$, where $\lambda = 1/\mathrm{x02am}$.

2: **ak2 – double scalar**

The value of k^2.

Constraint: $0.0 \leq \mathbf{ak2} \leq 1.0$.

5.2 Optional Input Parameters

None.

5.3 Input Parameters Omitted from the MATLAB Interface

None.

5.4 Output Parameters

1: **sn – complex scalar**
2: **cn – complex scalar**
3: **dn – complex scalar**

The values of the functions $\mathrm{sn}\,z$, $\mathrm{cn}\,z$ and $\mathrm{dn}\,z$, respectively.

4: **ifail – int32 scalar**

0 unless the function detects an error (see Section 6).

6 Error Indicators and Warnings

Errors or warnings detected by the function:

ifail $= 1$

On entry, $\mathbf{ak2} < 0.0$,
or $\mathbf{ak2} > 1.0$,
or $\mathrm{ABS}(\mathrm{Re}(\mathbf{z})) > \sqrt{\lambda}$,
or $\mathrm{ABS}(\mathrm{Im}(\mathbf{z})) > \sqrt{\lambda}$, where $\lambda = 1/\mathrm{x02am}$.

7 Accuracy

In principle the function is capable of achieving full relative precision in the computed values. However, the accuracy obtainable in practice depends on the accuracy of the standard elementary functions such as SIN and COS.

8 Further Comments

The values of $\operatorname{sn} z$, $\operatorname{cn} z$ and $\operatorname{dn} z$ are computed via the formulae

$$\begin{aligned}
\operatorname{sn} z &= \frac{\operatorname{sn}(u,k)\operatorname{dn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')} + i\frac{\operatorname{cn}(u,k)\operatorname{dn}(u,k)\operatorname{sn}(v,k')\operatorname{cn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')} \\
\operatorname{cn} z &= \frac{\operatorname{cn}(u,k)\operatorname{cn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')} - i\frac{\operatorname{sn}(u,k)\operatorname{dn}(u,k)\operatorname{sn}(v,k')\operatorname{dn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')} \\
\operatorname{dn} z &= \frac{\operatorname{dn}(u,k)\operatorname{cn}(v,k')\operatorname{dn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')} - i\frac{k^2\operatorname{sn}(u,k)\operatorname{cn}(u,k)\operatorname{sn}(v,k')}{1-\operatorname{dn}^2(u,k)\operatorname{sn}^2(v,k')},
\end{aligned}$$

where $z = u + iv$ and $k' = \sqrt{1-k^2}$ (the *complementary modulus*).

9 Example

```
z = complex(-2, +3);
ak2 = 0.25;
[sn, cn, dn, ifail] = s21cb(z, ak2)
```

```
sn =
  -1.5865 + 0.2456i
cn =
   0.3125 + 1.2468i
dn =
  -0.6395 - 0.1523i
ifail =
           0
```